

A Thom-Sebastiani Theorem in Characteristic p *

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Abstract

Let k be a perfect field of characteristic p , let $f_i : X_i \rightarrow \mathbb{A}_k^1$ ($i = 1, 2$) be two k -morphisms of finite type, and let $f : X_1 \times_k X_2 \rightarrow \mathbb{A}_k^1$ be the morphism defined by $f(z_1, z_2) = f_1(z_1) + f_2(z_2)$. For each $i \in \{1, 2\}$, let x_i be a k -rational point in the fiber $f_i^{-1}(0)$ such that f_i is smooth on $X_i - \{x_i\}$. Using the ℓ -adic Fourier transformation and the stationary phase principle of Laumon, we prove that the vanishing cycle of f at $x = (x_1, x_2)$ is the convolution product of the vanishing cycles of f_i at x_i ($i = 1, 2$).

Key words: vanishing cycle, nearby cycle, local Fourier transformation, perverse sheaf.

Mathematics Subject Classification: 14F20.

Introduction

Let $f_i : (\mathbb{C}^{n_i}, 0) \rightarrow (\mathbb{C}, 0)$ ($i = 1, 2$) be two germs of analytic functions with isolated critical points, and consider the germ $f : (\mathbb{C}^{n_1+n_2}, 0) \rightarrow (\mathbb{C}, 0)$ defined by $f(z_1, z_2) = f_1(z_1) + f_2(z_2)$. The classical Thom-Sebastiani Theorem ([9]) states that the monodromy of f on the vanishing cycle is isomorphic to the tensor product of those of f_1 and f_2 . Deligne ([1], unpublished) studies the variation of the monodromy, and realizes that the tensor product in the Thom-Sebastiani theorem should be the convolution product. Using the Fourier transformation, the formula

$$\int e^{itf} = \int e^{itf_1} \int e^{itf_2},$$

and the asymptotic expansions of such integrals for $t \rightarrow \infty$, Varchenko ([10]) proves a Thom-Sebastiani Theorem for the Hodge spectrum. In this paper, using the ℓ -adic Fourier transformation, we prove a Thom-Sebastiani theorem in characteristic p .

Throughout this paper, k is a perfect field of characteristic p , ℓ is a prime number distinct from p , and S is a henselian trait of equal characteristic p with generic point η and special point s such that $k(s) = k$. Let $X \rightarrow S$ be a morphism of finite type, and let K be an object in the category $D_c^b(X, \overline{\mathbb{Q}}_\ell)$

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constructed in [2, 1.1]. We refer the reader to [5] for the definitions and properties of the nearby cycle $R\Psi(K)$ and the vanishing cycle $R\Phi(K)$.

Lemma 0.1. *Let $X \rightarrow S$ be a morphism of finite type, let x be a k -rational point in the special fiber X_s , and let $K \in \text{ob } D_c^b(X, \overline{\mathbb{Q}}_\ell)$. Suppose $X - \{x\} \rightarrow S$ is smooth and $\mathcal{H}^q(K)|_{X-\{x\}}$ are lisse for all q .*

(i) $R\Phi(K)|_{X_{\bar{s}}-\{x\}} = 0$.

(ii) Suppose furthermore that X is pure of dimension n and regular at x , and K is a lisse sheaf. Then $R^i\Phi(K)$ vanishes for $i \neq n-1$, and $R^{n-1}\Phi(K)$ is a skyscraper sheaf on $X_{\bar{s}}$ supported at x .

Proof. (i) follows from the smooth base change theorem. Under the assumption of (ii), $K[n]$ is a perverse sheaf on X . By [7, 4.6], $R\Phi(K[n])[-1]$ is perverse. Combined with (i), we see that $R\Phi(K[n])[-1]$ is a perverse sheaf on $X_{\bar{s}}$ supported at x . Our assertion follows. \square

Let $\mathbb{A}_{(0)}^1$ be the henselization of \mathbb{A}_k^1 at 0, let η_0 be its generic point, and let $j : \eta_0 \hookrightarrow \mathbb{A}_{(0)}^1$ be the canonical open immersion. We can identify a $\text{Gal}(\bar{\eta}_0/\eta_0)$ -module with a sheaf on η_0 . Let $(\mathbb{A}_k^1 \times_k \mathbb{A}_k^1)_{(0,0)}$ be the henselization of $\mathbb{A}_k^1 \times_k \mathbb{A}_k^1$ at $(0,0)$, and let

$$\tilde{p}_1, \tilde{p}_2, \tilde{a} : (\mathbb{A}_k^1 \times_k \mathbb{A}_k^1)_{(0,0)} \rightarrow \mathbb{A}_{(0)}^1$$

be the morphisms induced by the two projections $p_1, p_2 : \mathbb{A}_k^1 \times_k \mathbb{A}_k^1 \rightarrow \mathbb{A}_k^1$ and the addition $a : \mathbb{A}_k^1 \times_k \mathbb{A}_k^1 \rightarrow \mathbb{A}_k^1$ of the algebraic group \mathbb{A}_k^1 , respectively. Let V_1 and V_2 be $\overline{\mathbb{Q}}_\ell$ -representations of $\text{Gal}(\bar{\eta}_0/\eta_0)$, and regard them as sheaves on η_0 . By [8, 2.7.1.3], the vanishing cycle $R\Phi_{\eta_0}(\tilde{p}_1^* j! V_1 \otimes^L \tilde{p}_2^* j! V_2)$ relative to the morphism \tilde{a} is nonzero only at $(0,0)$ and at degree 1. We define the convolution product of V_1 and V_2 to be the $\text{Gal}(\bar{\eta}_0/\eta_0)$ -module

$$V_1 * V_2 = R^1\Phi_{\eta_0}(\tilde{p}_1^* j! V_1 \otimes \tilde{p}_2^* j! V_2)_{(0,0)}.$$

More generally, if V_1 and V_2 are objects in $D_c^b(\eta_0, \overline{\mathbb{Q}}_\ell)$, we define their convolution product to be

$$V_1 * V_2 = R\Phi_{\eta_0}(\tilde{p}_1^* j! V_1 \otimes^L \tilde{p}_2^* j! V_2)_{(0,0)}$$

Our main result is the following:

Theorem 0.2. *Let $f_i : X_i \rightarrow \mathbb{A}_k^1$ ($i = 1, 2$) be two k -morphisms of finite type, $K_i \in \text{ob } D_c^b(X_i, \overline{\mathbb{Q}}_\ell)$, $X = X_1 \times_S X_2$, $K = K_1 \boxtimes^L K_2$, and $f : X = X_1 \times_k X_2 \rightarrow \mathbb{A}_k^1$ the morphism defined by*

$$f(z_1, z_2) = f_1(z_1) + f_2(z_2).$$

For each $i \in \{1, 2\}$, let x_i be a k -rational point in the fiber $f_i^{-1}(0)$. Suppose $\mathcal{H}^q(K_i)|_{X_i - \{x_i\}}$ are lisse for all q and $f_i|_{X_i - \{x_i\}}$ is smooth. Denote by x the k -rational point (x_1, x_2) on X . Denote respectively by $R\Phi_X$ and $R\Phi_{X_i}$ the vanishing cycle functors relative to the morphisms

$$X \times_{\mathbb{A}_k^1} \mathbb{A}_{(0)}^1 \rightarrow \mathbb{A}_{(0)}^1, \quad X_i \times_{\mathbb{A}_k^1} \mathbb{A}_{(0)}^1 \rightarrow \mathbb{A}_{(0)}^1$$

obtained from f and f_i by base change.

(i) x is the only point in $f^{-1}(0)$ where f is not smooth. As objects in $D_c^b(\eta_0, \overline{\mathbb{Q}}_\ell)$, we have a canonical isomorphism

$$(R\Phi_{X_1}(K_1))_{x_1} * (R\Phi_{X_2}(K_2))_{x_2} \cong (R\Phi_X(K))_x.$$

(ii) Suppose furthermore that for each i , X_i is regular at x_i and K_i is a lisse sheaf. Let $n_i = \dim \mathcal{O}_{X_i, x_i}$, and let $n = n_1 + n_2$. Then X is regular at x , and as $\text{Gal}(\bar{\eta}_0/\eta_0)$ -modules, we have a canonical isomorphism

$$(R^{n_1-1}\Phi_{X_1}(K_1))_{x_1} * (R^{n_2-1}\Phi_{X_2}(K_2))_{x_2} \cong (R^{n-1}\Phi_X(K))_x.$$

The Artin-Schreier morphism

$$\mathbb{A}_k^1 \rightarrow \mathbb{A}_k^1, \quad t \mapsto t^p - t$$

is a \mathbb{Z}/p -torsor. Fix a nontrivial additive character $\psi : \mathbb{Z}/p \rightarrow \overline{\mathbb{Q}}_\ell^*$. Pushing-forward the Artin-Schreier torsor using ψ^{-1} , we get a lisse sheaf \mathcal{L}_ψ on \mathbb{A}_k^1 . Denote the inverse image of \mathcal{L}_ψ under the morphism

$$\mathbb{A}_k^1 \times_k \mathbb{A}_k^1 \rightarrow \mathbb{A}_k^1, \quad (t, t') \rightarrow tt'$$

by $\mathcal{L}_\psi(tt')$. Let $\mathbb{A}_k^1 \times_k \mathbb{A}_k^1 \hookrightarrow \mathbb{A}_k^1 \times_k \mathbb{P}_k^1$ be the open immersion defined by the canonical open immersion $\mathbb{A}_k^1 = \mathbb{P}_k^1 - \{\infty\} \hookrightarrow \mathbb{P}_k^1$. Denote by $\overline{\mathcal{L}_\psi(tt')}$ the sheaf on $\mathbb{A}_k^1 \times_k \mathbb{P}_k^1$ obtained from the sheaf $\mathcal{L}_\psi(tt')$ on $\mathbb{A}_k^1 \times_k \mathbb{A}_k^1$ by extension by zero. To distinguish the two factors in $\mathbb{A}_k^1 \times_k \mathbb{A}_k^1$ and in $\mathbb{A}_k^1 \times_k \mathbb{P}_k^1$, we denote objects related to the second factor by symbols with the supcript $'$. Denote by $\mathbb{P}_{(\infty')}^1$ the henselization of \mathbb{P}_k^1 at ∞' , denote by $\eta_{\infty'}$ its generic point, and denote the restriction of $\overline{\mathcal{L}_\psi(tt')}$ to $\mathbb{A}_{(0)}^1 \times_k \mathbb{P}_{(\infty')}^1$ also by $\overline{\mathcal{L}_\psi(tt')}$. Fix a uniformizer π of S . We have a k -morphism $S \rightarrow \mathbb{A}_k^1$ induced by the k -homomorphism

$$k[t] \rightarrow \Gamma(S, \mathcal{O}_S), \quad t \mapsto \pi.$$

It induces a k -morphism $S \rightarrow \mathbb{A}_{(0)}^1$ which we denote also by π . Denote by $\overline{\mathcal{L}_\psi(\pi t')}$ the inverse image of $\overline{\mathcal{L}_\psi(tt')}$ under the morphism

$$S \times_k \mathbb{P}_{(\infty')}^1 \xrightarrow{\pi \times \text{id}_{\mathbb{P}_{(\infty')}^1}} \mathbb{A}_{(0)}^1 \times_k \mathbb{P}_{(\infty')}^1.$$

Our proof of Theorem 0.2 relies on the following lemma:

Lemma 0.3. *Let $g : Y \rightarrow S$ be a morphism of finite type, let $K \in \text{ob } D_c^b(Y, \overline{\mathbb{Q}}_\ell)$, and let y be a k -rational point in the special fiber $g^{-1}(s)$. Suppose $g|_{Y-\{y\}}$ is smooth, and $\mathcal{H}^q(K)|_{Y-\{y\}}$ are lisse for all q .*

(i) *$R\Phi_{\eta_{\infty'}}(p_1^* j_!(R\Phi_Y(K)_y) \otimes^L \overline{\mathcal{L}_\psi(\pi t')})$ is supported at $(s, \overline{\infty}')$, and $R\Phi_{\eta_{\infty'}}(\text{pr}_1^* K \otimes^L (g \times \text{id}_{\mathbb{P}_{(\infty')}^1})^* \overline{\mathcal{L}_\psi(\pi t')})$ is supported at $(y, \overline{\infty}')$, where $R\Phi_Y$ denotes the vanishing cycle functor for the morphism g , $R\Phi_Y(K)_y$ is a complex of $\text{Gal}(\bar{\eta}/\eta)$ -module and is regarded as an object on $D_c^b(\eta, \overline{\mathbb{Q}}_\ell)$, $j : \eta \hookrightarrow S$ is the canonical open immersion, $R\Phi_{\eta_{\infty'}}$ denotes the vanishing cycle functors for the projections*

$$S \times_k \mathbb{P}_{(\infty')}^1 \rightarrow \mathbb{P}_{(\infty')}^1, \quad Y \times_k \mathbb{P}_{(\infty')}^1 \rightarrow \mathbb{P}_{(\infty')}^1,$$

and p_1, pr_1 are the projections

$$p_1 : S \times_k \mathbb{P}_{(\infty')}^1 \rightarrow S, \quad \text{pr}_1 : Y \times_k \mathbb{P}_{(\infty')}^1 \rightarrow Y.$$

(ii) *We have a canonical isomorphism*

$$R\Phi_{\eta_{\infty'}}(p_1^* j_!(R\Phi_Y(K)_y) \otimes^L \overline{\mathcal{L}_\psi(\pi t')})_{(s, \infty')} \cong R\Phi_{\eta_{\infty'}}(\text{pr}_1^* K \otimes^L (g \times \text{id}_{\mathbb{P}_{(\infty')}^1})^* \overline{\mathcal{L}_\psi(\pi t')})_{(y, \infty')}.$$

(iii) *Suppose furthermore that Y is pure of dimension n and regular at y , and K is a lisse sheaf. Then $R^i \Phi_{\eta_{\infty'}}(\text{pr}_1^* K \otimes^L (g \times \text{id}_{\mathbb{P}_{(\infty')}^1})^* \overline{\mathcal{L}_\psi(\pi t')})$ vanishes for $i \neq n$, and $R^n \Phi_{\eta_{\infty'}}(\text{pr}_1^* K \otimes^L (g \times \text{id}_{\mathbb{P}_{(\infty')}^1})^* \overline{\mathcal{L}_\psi(\pi t')})$ is a skyscraper sheaf on $Y \times_k \overline{\infty}'$ supported at $(y, \overline{\infty}')$.*

(iv) *Under the condition of (iii), we have a canonical isomorphism of $\text{Gal}(\bar{\eta}_{\infty'}/\eta_{\infty'})$ -modules*

$$\mathcal{F}^{(0, \infty')} (R^{n-1} \Phi_Y(K)_y) \cong R^n \Phi_{\eta_{\infty'}}(\text{pr}_1^* K \otimes^L (g \times \text{id}_{\mathbb{P}_{(\infty')}^1})^* \overline{\mathcal{L}_\psi(\pi t')})_{(y, \infty')},$$

where $\mathcal{F}^{(0, \infty')}$ is the local Fourier transformation defined by Laumon ([8, 2.4.2.3]).

We will prove Lemma 0.3 in §1. Let's deduce Theorem 0.2 from Lemma 0.3 and the Künneth formula for nearby cycles in [7, 4.7].

Proof of Theorem 0.2. It is not hard to check that x is the only point in $f^{-1}(0)$ where f is not smooth, and X is regular at x if X_i are regular at x_i . By [3, 1.3.1], we have a canonical isomorphism

$$(f_1 \times \text{id}_{\mathbb{P}_k^1})^* \overline{\mathcal{L}_\psi(tt')} \boxtimes^L (f_2 \times \text{id}_{\mathbb{P}_k^1})^* \overline{\mathcal{L}_\psi(tt')} \cong (f \times \text{id}_{\mathbb{P}_k^1})^* \overline{\mathcal{L}_\psi(tt')}.$$

So we have

$$(\text{pr}_{X_1}^* K_1 \otimes^L (f_1 \times \text{id}_{\mathbb{P}_k^1})^* \overline{\mathcal{L}_\psi(tt')}) \boxtimes^L (\text{pr}_{X_2}^* K_2 \otimes^L (f_2 \times \text{id}_{\mathbb{P}_k^1})^* \overline{\mathcal{L}_\psi(tt')}) \cong \text{pr}_X^* K \otimes^L (f \times \text{id}_{\mathbb{P}_k^1})^* \overline{\mathcal{L}_\psi(tt')},$$

where

$$\mathrm{pr}_X : X_1 \times_k X_2 \times_k \mathbb{P}_k^1 \rightarrow X_1 \times_k X_2, \quad \mathrm{pr}_{X_1} : X_1 \times_k \mathbb{P}_k^1 \rightarrow X_1, \quad \mathrm{pr}_{X_2} : X_2 \times_k \mathbb{P}_k^1 \rightarrow X_2$$

are the projections. By [7, 4.7], we have a canonical isomorphism

$$\begin{aligned} & R\Psi_{\eta_{\infty'}} \left(\mathrm{pr}_{X_1}^* K_1 \otimes^L (f_1 \times \mathrm{id}_{\mathbb{P}_k^1})^* \overline{\mathcal{L}_{\psi}(tt')} \right) \boxtimes^L R\Psi_{\eta_{\infty'}} \left(\mathrm{pr}_{X_2}^* K_2 \otimes^L (f_2 \times \mathrm{id}_{\mathbb{P}_k^1})^* \overline{\mathcal{L}_{\psi}(tt')} \right) \\ & \cong R\Psi_{\eta_{\infty'}} \left(\mathrm{pr}_X^* K \otimes^L (f \times \mathrm{id}_{\mathbb{P}_k^1})^* \overline{\mathcal{L}_{\psi}(tt')} \right). \end{aligned}$$

Since $\mathrm{pr}_X^* K \otimes^L (f \times \mathrm{id}_{\mathbb{P}_k^1})^* \overline{\mathcal{L}_{\psi}(tt')}$ vanishes on $X \times_k \infty'$, we have

$$R\Phi_{\eta_{\infty'}} \left(\mathrm{pr}_X^* K \otimes^L (f \times \mathrm{id}_{\mathbb{P}_k^1})^* \overline{\mathcal{L}_{\psi}(tt')} \right) \cong R\Psi_{\eta_{\infty'}} \left(\mathrm{pr}_X^* K \otimes^L (f \times \mathrm{id}_{\mathbb{P}_k^1})^* \overline{\mathcal{L}_{\psi}(tt')} \right),$$

and we have similar isomorphisms if we replace f by f_i and $\mathrm{pr}_X^* K$ by $\mathrm{pr}_{X_i}^* K_i$. So we have a canonical isomorphism

$$\begin{aligned} & R\Phi_{\eta_{\infty'}} \left(\mathrm{pr}_{X_1}^* K_1 \otimes^L (f_1 \times \mathrm{id}_{\mathbb{P}_k^1})^* \overline{\mathcal{L}_{\psi}(tt')} \right) \boxtimes^L R\Phi_{\eta_{\infty'}} \left(\mathrm{pr}_{X_2}^* K_2 \otimes^L (f_2 \times \mathrm{id}_{\mathbb{P}_k^1})^* \overline{\mathcal{L}_{\psi}(tt')} \right) \\ & \cong R\Phi_{\eta_{\infty'}} \left(\mathrm{pr}_X^* K \otimes^L (f \times \mathrm{id}_{\mathbb{P}_k^1})^* \overline{\mathcal{L}_{\psi}(tt')} \right). \end{aligned}$$

Combined with Lemma 0.3 (i), we get

$$\begin{aligned} & R\Phi_{\eta_{\infty'}} \left(\mathrm{pr}_{X_1}^* K_1 \otimes^L (f_1 \times \mathrm{id}_{\mathbb{P}_k^1})^* \overline{\mathcal{L}_{\psi}(tt')} \right)_{(x_1, \infty')} \otimes^L R\Phi_{\eta_{\infty'}} \left(\mathrm{pr}_{X_2}^* K_2 \otimes^L (f_2 \times \mathrm{id}_{\mathbb{P}_k^1})^* \overline{\mathcal{L}_{\psi}(tt')} \right)_{(x_2, \infty')} \\ & \cong R\Phi_{\eta_{\infty'}} \left(\mathrm{pr}_X^* K \otimes^L (f \times \mathrm{id}_{\mathbb{P}_k^1})^* \overline{\mathcal{L}_{\psi}(tt')} \right)_{(x, \infty')}. \end{aligned}$$

By Lemma 0.3 (ii), this induces a canonical isomorphism

$$\begin{aligned} & R\Phi_{\eta_{\infty'}} \left(p_{1,j}^* (R\Phi_{X_1}(K_1)_{x_1}) \otimes^L \overline{\mathcal{L}_{\psi}(tt')} \right)_{(0, \infty')} \otimes^L R\Phi_{\eta_{\infty'}} \left(p_{1,j}^* (R\Phi_{X_2}(K_2)_x) \otimes^L \overline{\mathcal{L}_{\psi}(tt')} \right)_{(0, \infty')} \\ (1) \quad & \cong R\Phi_{\eta_{\infty'}} \left(p_{1,j}^* (R\Phi_X(K)_x) \otimes^L \overline{\mathcal{L}_{\psi}(tt')} \right)_{(0, \infty')}, \end{aligned}$$

and under the assumption of Theorem 0.2 (ii), we get a canonical isomorphism

$$\begin{aligned} & \mathcal{F}^{(0, \infty')} \left(R^{n_1-1} \Phi_{X_1}(K_1)_{x_1} \right) \otimes \mathcal{F}^{(0, \infty')} \left(R^{n_2-1} \Phi_{X_2}(K_2)_{x_2} \right) \\ & \cong \mathcal{F}^{(0, \infty')} \left(R^{n-1} \Phi_X(K)_x \right). \end{aligned}$$

By [8, 2.7.2.2 (i)] and the inversion formula for local Fourier transformation [8, 2.4.3 (i) c)], under the assumption of Theorem 0.2 (ii), we have a canonical isomorphism

$$R^{n_1-1} \Phi_{X_1}(K_1)_{x_1} * R^{n_2-1} \Phi_{X_2}(K_2)_{x_2} \cong R^{n-1} \Phi_X(K)_x.$$

The argument in the proof of various results in [8] also shows that in general we have a canonical isomorphism

$$R\Phi_{X_1}(K_1)_{x_1} * R\Phi_{X_2}(K_2)_{x_2} \cong R\Phi_X(K)_x.$$

We give a detailed argument for completeness. Choose $L, L_1, L_2 \in \text{ob } D_c^b(\mathbb{A}_k^1 - \{0\}, \overline{\mathbb{Q}}_\ell)$ such that

$$L|_{\eta_0} \cong R\Phi_X(K)_x, \quad L_1|_{\eta_0} \cong R\Phi_{X_1}(K_1)_{x_1}, \quad L_2|_{\eta_0} \cong R\Phi_{X_2}(K_2)_{x_2},$$

and such that $\mathcal{H}^q(L), \mathcal{H}^q(L_1), \mathcal{H}^q(L_2)$ are lisse on $\mathbb{A}_k^1 - \{0\}$ and tamely ramified at ∞ for all q .

Let

$$\mathcal{F} : D_c^b(\mathbb{A}_k^1, \overline{\mathbb{Q}}_\ell) \rightarrow D_c^b(\mathbb{A}_k^1, \overline{\mathbb{Q}}_\ell)$$

be the Fourier-Deligne transformation (confer [8, 1.2.1.1]), and let $\iota : \mathbb{A}_k^1 - \{0\} \hookrightarrow \mathbb{A}_k^1$ be the canonical open immersion. By [8, 2.3.3.1 (iii), 2.4.3 (iii) b)], we have

$$\begin{aligned} \mathcal{F}(\iota_! L)[-1]|_{\eta_{\infty'}} &\cong R\Phi_{\eta_{\infty'}} \left(p_1^* j_! (R\Phi_X(K)_x) \otimes^L \overline{\mathcal{L}_\psi(tt')} \right)_{(0, \infty')}, \\ \mathcal{F}(\iota_! L_1)[-1]|_{\eta_{\infty'}} &\cong R\Phi_{\eta_{\infty'}} \left(p_1^* j_! (R\Phi_{X_1}(K_1)_{x_1}) \otimes^L \overline{\mathcal{L}_\psi(tt')} \right)_{(0, \infty')}, \\ \mathcal{F}(\iota_! L_2)[-1]|_{\eta_{\infty'}} &\cong R\Phi_{\eta_{\infty'}} \left(p_1^* j_! (R\Phi_{X_2}(K_2)_{x_2}) \otimes^L \overline{\mathcal{L}_\psi(tt')} \right)_{(0, \infty')}. \end{aligned}$$

So we can write the isomorphism (1) as

$$(\mathcal{F}(\iota_! L_1) \otimes \mathcal{F}(\iota_! L_2))[-2]|_{\eta_{\infty'}} \cong \mathcal{F}(\iota_! L)[-1]|_{\eta_{\infty'}}.$$

By [8, 1.2.2.7], we get the isomorphism

$$\mathcal{F}(\iota_! L_1 * \iota_! L_2)|_{\eta_{\infty'}} \cong \mathcal{F}(\iota_! L)|_{\eta_{\infty'}},$$

where $\iota_! L_1 * \iota_! L_2$ is the (global) convolution product of $\iota_! L_1$ and $\iota_! L_2$ (confer [8, 1.2.2.6]). By the above isomorphism and Lemma 0.4 below, we have

$$R\Phi_{\eta_0}(\iota_! L_1 * \iota_! L_2) \cong R\Phi_{\eta_0}(\iota_! L),$$

where $R\Phi_{\eta_0}$ denotes the vanishing cycle functor on \mathbb{A}_k^1 . By [8, 2.7.1.1 (iii)], this last isomorphism is exactly

$$R\Phi_{X_1}(K_1)_{x_1} * R\Phi_{X_2}(K_2)_{x_2} \cong R\Phi_X(K)_x.$$

□

Lemma 0.4. *Let $L_1, L_2 \in \text{ob } D_c^b(\mathbb{A}_k^1, \overline{\mathbb{Q}}_\ell)$. If $\mathcal{F}(L_1)|_{\eta_{\infty'}} \cong \mathcal{F}(L_2)|_{\eta_{\infty'}}$, then $R\Phi_{\eta_0}(L_1) \cong R\Phi_{\eta_0}(L_2)$.*

Proof. Let $L \in \text{ob } D_c^b(\mathbb{A}_k^1, \overline{\mathbb{Q}}_\ell)$ and let $L' = \mathcal{F}(L)$. By the inversion formula for the Fourier-Deligne transformation [8, 1.2.2.1], we have

$$\mathcal{F}'(L') = b_* L(-1),$$

where $b : \mathbb{A}_k^1 \rightarrow \mathbb{A}_k^1$ is the morphism defined by $t \mapsto -t$, and \mathcal{F}' denote the Fourier-Deligne transformation for the dual of \mathbb{A}_k^1 (which is the second factor of $\mathbb{A}_k^1 \times_k \mathbb{A}_k^1$). So we have

$$R\Phi_{\eta_0}(L) = b^* R\Phi_{\eta_0}(\mathcal{F}'(L'))(1).$$

By [8, 2.3.2.1], $R\Phi_{\eta_0}(\mathcal{F}'(L'))$ depends only on $L'|_{\eta_{\infty'}} = \mathcal{F}(L)|_{\eta_{\infty'}}$. Our assertion follows. \square

Deligne suggests to me the following more general Thom-Sebastiani problem. Let Λ be a noetherian torsion ring with the property $m\Lambda = 0$ for some integer m relatively prime to p . let S_1, S_2, S be henselian traits over $\text{Spec } k$ of equal characteristic, all with residue field k , let s_1, s_2, s be their closed points, and let $a : S_1 \times_k S_2 \rightarrow S$ be a k -morphism such that $a(s_1, s_2) = s$ and such that $a(\cdot, s_2) : S_1 \rightarrow S$ and $a(s_1, \cdot) : S_2 \rightarrow S$ are isomorphisms. For each i , let $f_i : X_i \rightarrow S_i$ be a morphism of finite type, let x_i be a k -rational point in the special fiber such that $f_i|_{X_i - x_i}$ is smooth, and let $K_i \in \text{ob } D_{ctf}^b(X_i, \mathbb{Z}/\ell^m)$ such that $\mathcal{H}^q(K_i)|_{X_i - x_i}$ are locally constant for all q . Let η_i be the generic point of S_i , and let $j_i : \eta_i \hookrightarrow S_i$ be the canonical open immersion. Deligne makes the following conjecture.

Conjecture 0.5.

(i) Under the above conditions, $a \circ (f_1 \times f_2)$ is locally acyclic relative to $K_1 \boxtimes^L K_2$ outside (x_1, x_2) , and hence the vanishing cycle $R\Phi_{X_1 \times_k X_2}(K_1 \boxtimes^L K_2)$ relative to the morphism $a \circ (f_1 \times f_2)$ is supported at (x_1, x_2) . (Confer [4, 2.12] for the definition of local acyclicity.)

(ii) There is a canonical isomorphism

$$\begin{aligned} & R\Phi_{S_1 \times_k S_2} \left(j_{1!}((R\Phi_{X_1}(K_1))_{x_1}) \boxtimes^L j_{2!}((R\Phi_{X_2}(K_2))_{x_2}) \right)_{(s_1, s_2)} \\ & \cong \left(R\Phi_{X_1 \times_k X_2}(K_1 \boxtimes^L K_2) \right)_{(x_1, x_2)}, \end{aligned}$$

where $R\Phi_{S_1 \times_k S_2}$ is the vanishing cycle functor for the morphism a , and $(R\Phi_{X_i}(K_i))_{x_i}$ ($i = 1, 2$) are complexes of Λ -modules with $\text{Gal}(\bar{\eta}_i/\eta_i)$ -action and are regarded as objects in $D_{ctf}^b(\eta_i, \Lambda)$.

Fix notation by the following commutative diagram:

$$\begin{array}{ccccc}
(X_1 \times_k X_2)_{(x_1, x_2)} & \rightarrow & X_{1(x_1)} \times_k X_{2(x_2)} & \rightarrow & X_1 \times_k X_2 \\
\downarrow (f_1 \times f_2) & & \downarrow f_1 \times f_2 & \swarrow f_1 \times f_2 & \\
(S_1 \times_k S_2)_{(s_1, s_2)} & \rightarrow & S_1 \times_k S_2 & & \\
& \searrow \tilde{a} & \downarrow a & & \\
& & S & &
\end{array} ,$$

where a notation like $X_{(x)}$ means the henselization of X at x , and a notation like \tilde{f} means the morphism on henselizations induced by f . Let $\tilde{p}_i : (S_1 \times_k S_2)_{(s_1, s_2)} \rightarrow S_i$ ($i = 1, 2$) be the morphism induced by the projection $S_1 \times_k S_2 \rightarrow S_i$. We have a canonical morphism

$$\tilde{p}_1^* R\tilde{f}_{1*} K_1 \otimes^L \tilde{p}_2^* R\tilde{f}_{2*} K_2 \rightarrow R(\widetilde{f_1 \times f_2})_*(K_1 \boxtimes^L K_2),$$

where we denote the restrictions of $K_1 \boxtimes^L K_2$, K_1 and K_2 to $(X_1 \times_k X_2)_{(x_1, x_2)}$, $X_{1(x_1)}$ and $X_{2(x_2)}$ respectively by the same notation. It induces a canonical morphism

$$(2) \quad R\Phi\left(R\tilde{a}_*(\tilde{p}_1^* R\tilde{f}_{1*} K_1 \otimes^L \tilde{p}_2^* R\tilde{f}_{2*} K_2)\right) \rightarrow R\Phi\left(R\tilde{a}_* R(\widetilde{f_1 \times f_2})(K_1 \boxtimes^L K_2)\right).$$

Using the construction in the proof of Lemma 1.1 in §1, one can construct a canonical morphism

$$\begin{aligned}
(3) \quad & R\Phi\left(R\tilde{a}_*(\tilde{p}_1^* R\tilde{f}_{1*} K_1 \otimes^L \tilde{p}_2^* R\tilde{f}_{2*} K_2)\right) \\
& \rightarrow R\Phi\left(R\tilde{a}_*(\tilde{p}_1^* j_{1!} R\Phi(R\tilde{f}_{1*} K_1) \otimes^L \tilde{p}_2^* j_{2!} R\Phi(R\tilde{f}_{2*} K_2))\right).
\end{aligned}$$

By Lemma 1.2 in §1, we have canonical isomorphisms

$$\begin{aligned}
(4) \quad & R\Phi\left(R\tilde{a}_* R(\widetilde{f_1 \times f_2})(K_1 \boxtimes^L K_2)\right) \\
& \cong R\Phi_{X_1 \times_k X_2}(K_1 \boxtimes^L K_2)_{(x_1, x_2)},
\end{aligned}$$

$$\begin{aligned}
(5) \quad & R\Phi\left(R\tilde{a}_*(\tilde{p}_1^* j_{1!} R\Phi(R\tilde{f}_{1*} K_1) \otimes^L \tilde{p}_2^* j_{2!} R\Phi(R\tilde{f}_{2*} K_2))\right) \\
& \cong R\Phi_{S_1 \times_k S_2}\left(j_{1!}((R\Phi_{X_1}(K_1))_{x_1}) \boxtimes^L j_{2!}((R\Phi_{X_2}(K_2))_{x_2})\right)_{(s_1, s_2)}.
\end{aligned}$$

If one can prove that the canonical morphisms (2) and (3) are isomorphisms, then taking the composite $(4) \circ (2) \circ (3)^{-1} \circ (5)^{-1}$, one should get the canonical isomorphism in the conjecture. Due to the use of the Fourier transformation, the method used in this paper is not applicable to the proof of the conjecture unless a is directly related to the addition of the algebraic group \mathbb{A}_k^1 .

1 Proof of Lemma 0.3

We first prove Lemma 0.3 (i) and (iii).

Proof of Lemma 0.3 (i) and (iii). The fact that $R\Phi_{\eta_{\infty'}} \left(p_1^* j_! (R\Phi_Y(K)_y) \otimes^L \overline{\mathcal{L}_\psi(\pi t')} \right)$ is supported at $(s, \overline{\infty'})$ follows from [8, 2.4.2.2]. By [8, 1.3.1.2], $R\Phi_{\eta_{\infty'}} \left(\overline{\mathcal{L}_\psi(\pi t')} \right)$ vanishes on $S \times_k \overline{\infty'}$. On the other hand, $g \times \text{id}_{\mathbb{P}_{(\infty')}^1}$ is smooth on $(Y - \{y\}) \times_k \mathbb{P}_{(\infty')}^1$, and $\mathcal{H}^q(K)$ are lisse on $Y - \{y\}$ for all q . By the smooth base change theorem and the projection formula, we have

$$\begin{aligned} & R\Phi_{\eta_{\infty'}} \left(\text{pr}_1^* K \otimes^L (g \times \text{id}_{\mathbb{P}_{(\infty')}^1})^* \overline{\mathcal{L}_\psi(\pi t')} \right) |_{(Y - \{y\}) \times_k \overline{\infty'}} \\ & \cong (\text{pr}_1^* K) |_{(Y - \{y\}) \times_k \overline{\infty'}} \otimes^L \left((g \times \text{id}_{\mathbb{P}_{(\infty')}^1})^* R\Phi_{\eta_{\infty'}} \left(\overline{\mathcal{L}_\psi(\pi t')} \right) \right) |_{(Y - \{y\}) \times_k \overline{\infty'}} \\ & = 0. \end{aligned}$$

It follows that $R\Phi_{\eta_{\infty'}} \left(\text{pr}_1^* K \otimes^L (g \times \text{id}_{\mathbb{P}_{(\infty')}^1})^* \overline{\mathcal{L}_\psi(\pi t')} \right)$ is supported at $(y, \overline{\infty'})$. This proves Lemma 0.3 (i).

Note that the restriction of $\overline{\mathcal{L}_\psi(\pi t')}$ to $S \times_k \eta_{\infty'}$ is a lisse sheaf. So under the assumption of (iii), the restriction of $\text{pr}_1^* K \otimes^L (g \times \text{id}_{\mathbb{P}_{(\infty')}^1})^* \overline{\mathcal{L}_\psi(\pi t')}[n]$ to $Y \times_k \eta_{\infty'}$ is perverse. By [7, 4.5], $R\Psi_{\eta_{\infty'}} \left(\text{pr}_1^* K \otimes^L (g \times \text{id}_{\mathbb{P}_{(\infty')}^1})^* \overline{\mathcal{L}_\psi(\pi t')} \right)[n]$ is perverse on $Y \times_k \overline{\infty'}$. Note that $\text{pr}_1^* K \otimes^L (g \times \text{id}_{\mathbb{P}_{(\infty')}^1})^* \overline{\mathcal{L}_\psi(\pi t')}$ vanishes on $Y \times_k \overline{\infty'}$. So we have

$$R\Phi_{\eta_{\infty'}} \left(\text{pr}_1^* K \otimes^L (g \times \text{id}_{\mathbb{P}_{(\infty')}^1})^* \overline{\mathcal{L}_\psi(\pi t')} \right) \cong R\Psi_{\eta_{\infty'}} \left(\text{pr}_1^* K \otimes^L (g \times \text{id}_{\mathbb{P}_{(\infty')}^1})^* \overline{\mathcal{L}_\psi(\pi t')} \right).$$

Hence $R\Phi_{\eta_{\infty'}} \left(\text{pr}_1^* K \otimes^L (g \times \text{id}_{\mathbb{P}_{(\infty')}^1})^* \overline{\mathcal{L}_\psi(\pi t')} \right)[n]$ is a perverse sheaf on $Y \times_k \overline{\infty'}$ supported at $(y, \overline{\infty'})$. Lemma 0.3 (iii) then follows. \square

Lemma 1.1. *Let K be an object in $D_c^b(S, \overline{\mathbb{Q}_\ell})$, let*

$$p_1 : S \times_k \mathbb{P}_{(\infty')}^1 \rightarrow S, \quad p_2 : S \times_k \mathbb{P}_{(\infty')}^1 \rightarrow \mathbb{P}_{(\infty')}^1$$

be the projections, and let $j : \eta \hookrightarrow S$ be the canonical open immersion. We have a canonical isomorphism

$$R\Phi_{\eta_{\infty'}} \left(p_1^* K \otimes^L \overline{\mathcal{L}_\psi(\pi t')} \right) \cong R\Phi_{\eta_{\infty'}} \left(p_1^* (j_! R\Phi_\eta(K)) \otimes^L \overline{\mathcal{L}_\psi(\pi t')} \right),$$

where $R\Phi_{\eta_{\infty'}}$ denotes the vanishing cycle functor with respect to the projection p_2 , $R\Phi_\eta(K)$ denotes the vanishing cycle of K which is a complex of $\text{Gal}(\overline{\eta}/\eta)$ -modules and is regarded as an object on $D_c^b(\eta, \overline{\mathbb{Q}_\ell})$.

Proof. Let E be a finite extension of \mathbb{Q}_ℓ in $\overline{\mathbb{Q}_\ell}$, let R be the integral closure of \mathbb{Z}_ℓ in E , and let λ be a uniformizer of R . It suffices to prove the same assertion for any complex K of sheaves of $R/(\lambda^m)$ -modules for any positive integer m . For any scheme X , denote by $K(X, R/(\lambda^m))$ the triangulated

category of complexes of etale sheaves of $R/(\lambda^m)$ -modules on X with morphisms being homotopy classes of morphisms of complexes. Let \tilde{S} be the strict henselization of S at a geometric point \bar{s} over s . Fix notation by the following commutative diagram:

$$\begin{array}{ccccc} \bar{\eta} & \xrightarrow{\bar{j}} & \tilde{S} & \xleftarrow{\bar{i}} & \bar{s} \\ \downarrow & & \downarrow & & \downarrow \\ \eta & \xrightarrow{j} & S & \xleftarrow{i} & s. \end{array}$$

For any complex $K \in \text{ob } K(S, R/(\lambda^m))$, we have

$$\begin{aligned} R\Gamma(\bar{s}, R\Psi_\eta(K)) &= R\Gamma(\bar{s}, \bar{i}^* R\bar{j}_* \bar{j}^*(K|_{\tilde{S}})) \\ &\cong \Gamma(\bar{s}, \bar{i}^* \bar{j}_* \bar{j}^*(K|_{\tilde{S}})) \\ &\cong K_{\bar{\eta}}. \end{aligned}$$

So $R\Psi_\eta(K)$ corresponds to the complex $K_{\bar{\eta}}$ of $R/(\lambda^m)$ -modules with $\text{Gal}(\bar{\eta}/\eta)$ -action. Recall that $R\Phi_\eta(K)$ is the mapping cone of the canonical morphism $\bar{i}^*(K|_{\tilde{S}}) \rightarrow R\Psi_\eta(K)$.

Let I be the inertia subgroup of $\text{Gal}(\bar{\eta}/\eta)$, that is, the kernel of the canonical epimorphism $\text{Gal}(\bar{\eta}/\eta) \rightarrow \text{Gal}(\bar{s}/s)$. The functor

$$K \mapsto (i^*K, j^*K, i^*K \rightarrow i^*j_*j^*K)$$

defines an equivalence of categories from the category $K(S, R/(\lambda^m))$ to the category of triples

$$(F, G, F \rightarrow G^I),$$

where F is a complex of $R/(\lambda^m)$ -modules with $\text{Gal}(\bar{s}/s)$ -action which can be identified with an object in $K(s, R/(\lambda^m))$, G is a complex of $R/(\lambda^m)$ -modules with $\text{Gal}(\bar{\eta}/\eta)$ -action which can be identified with an object in $K(\eta, R/(\lambda^m))$, and $F \rightarrow G^I$ is a morphism in $K(s, R/(\lambda^m))$. Given an object K in $K(S, R/(\lambda^m))$, consider the triples

$$K' = (K_{\bar{s}}, K_{\bar{s}}, K_{\bar{s}} \xrightarrow{\text{id}} K_{\bar{s}}), \quad K = (K_{\bar{s}}, K_{\bar{\eta}}, K_{\bar{s}} \rightarrow K_{\bar{\eta}}^I).$$

Note that the second object is exactly the triple associated to K and hence is denoted by K . The first object corresponds to a complex of constant sheaves on S . We have a canonical morphism $K' \rightarrow K$. Let K'' be the mapping cone of $K' \rightarrow K$. Then the canonical morphism $j_!j^*K'' \rightarrow K''$ defines a quasi-isomorphism $j_!R\Phi_\eta(K) \rightarrow K''$. We thus have a distinguished triangle

$$K' \rightarrow K \rightarrow j_!R\Phi_\eta(K) \rightarrow$$

in the derived category $D(S, R/(\lambda^m))$. It gives rise to a distinguished triangle

$$R\Psi_{\eta_{\infty}'} \left(p_1^* K' \otimes^L \overline{\mathcal{L}_{\psi}(\pi t')} \right) \rightarrow R\Psi_{\eta_{\infty}'} \left(p_1^* K \otimes^L \overline{\mathcal{L}_{\psi}(\pi t')} \right) \rightarrow R\Psi_{\eta_{\infty}'} \left(p_1^* (j_! R\Phi_{\eta}(K)) \otimes^L \overline{\mathcal{L}_{\psi}(\pi t')} \right) \rightarrow,$$

which can be identified with a distinguished triangle

$$R\Phi_{\eta_{\infty}'} \left(p_1^* K' \otimes^L \overline{\mathcal{L}_{\psi}(\pi t')} \right) \rightarrow R\Phi_{\eta_{\infty}'} \left(p_1^* K \otimes^L \overline{\mathcal{L}_{\psi}(\pi t')} \right) \rightarrow R\Phi_{\eta_{\infty}'} \left(p_1^* (j_! R\Phi_{\eta}(K)) \otimes^L \overline{\mathcal{L}_{\psi}(\pi t')} \right) \rightarrow$$

because $\overline{\mathcal{L}_{\psi}(\pi t')}$ vanishes on $S \times_k \overline{\infty}'$. Since K' corresponds to a complex of constant sheaves on S , we have

$$R\Phi_{\eta_{\infty}'} \left(p_1^* K' \otimes^L \overline{\mathcal{L}_{\psi}(\pi t')} \right) = 0$$

by [8, 1.3.1.2]. We thus have

$$R\Phi_{\eta_{\infty}'} \left(p_1^* K \otimes^L \overline{\mathcal{L}_{\psi}(\pi t')} \right) \cong R\Phi_{\eta_{\infty}'} \left(p_1^* (j_! R\Phi_{\eta}(K)) \otimes^L \overline{\mathcal{L}_{\psi}(\pi t')} \right).$$

□

Lemma 1.2. *Let $f : X \rightarrow S$ a morphism, x a k -rational point in the special fiber $f^{-1}(s)$, $X_{(x)}$ the henselization of X at x , $f_{(x)} : X_{(x)} \rightarrow S$ the morphism induced by f , and $K \in \text{ob } D_c^b(X, \overline{\mathbb{Q}}_{\ell})$. Then we have a canonical isomorphism*

$$R\Phi(Rf_{(x)*}(K|_{X_{(x)}})) \cong (R\Phi(K))_x.$$

Proof. Let $X_{(\bar{x})}$ (resp. \tilde{S}) be the strict henselization of X (resp. S) at a geometric point \bar{x} (resp. \bar{s}) over x (resp. s). Since x is a k -rational point, we have $X_{(\bar{x})} \cong X_{(x)} \times_S \tilde{S}$. Let $f_{(\bar{x})} : X_{(\bar{x})} \rightarrow \tilde{S}$ be the morphism induced by f . It can be identified with the base change of $f_{(x)} : X_{(x)} \rightarrow S$. Fix notation by the following commutative diagram:

$$\begin{array}{ccccc} X_{(\bar{x})} \times_{\tilde{S}} \bar{\eta} & \rightarrow & X_{(\bar{x})} & \leftarrow & X_{(\bar{x})} \times_{\tilde{S}} \bar{s} \\ f_{(\bar{x}), \bar{\eta}} \downarrow & & \downarrow f_{(\bar{x})} & & \downarrow f_{(\bar{x}), \bar{s}} \\ \bar{\eta} & \xrightarrow{\bar{j}} & \tilde{S} & \xleftarrow{\bar{i}} & \bar{s}. \end{array}$$

For convenience, denote the restrictions of K to $X_{(x)}$, $X_{(\bar{x})}$ and $X_{(\bar{x})} \times_{\tilde{S}} \bar{\eta}$ also by K . Then $R\Phi(Rf_{(x)*}K)$ is the mapping cone of the canonical morphism

$$\bar{i}^* Rf_{(\bar{x})*} K \rightarrow \bar{i}^* R\bar{j}_* \bar{j}^* Rf_{(\bar{x})*} K.$$

We have

$$\begin{aligned}
R\Gamma(\bar{s}, \bar{i}^* Rf_{(\bar{x})*} K) &\cong R\Gamma(\tilde{S}, Rf_{(\bar{x})*} K) \\
&\cong R\Gamma(X_{(\bar{x})}, K), \\
R\Gamma(\bar{s}, \bar{i}^* R\bar{j}_* \bar{j}^* Rf_{(\bar{x})*} K) &\cong R\Gamma(\tilde{S}, R\bar{j}_* \bar{j}^* Rf_{(\bar{x})*} K) \\
&\cong R\Gamma(\bar{\eta}, \bar{j}^* Rf_{(\bar{x})*} K) \\
&\cong R\Gamma(\bar{\eta}, Rf_{(\bar{x}), \bar{\eta}*} K) \\
&\cong R\Gamma(X_{(\bar{x})} \times_{\bar{S}} \bar{\eta}, K).
\end{aligned}$$

It follows that $R\Phi(Rf_{(x)*} K)$ can be identified with the mapping cone of the canonical morphism

$$R\Gamma(X_{(\bar{x})}, K) \rightarrow R\Gamma(X_{(\bar{x})} \times_{\bar{S}} \bar{\eta}, K).$$

The later is exactly $(R\Phi(K))_x$. □

Under the assumption of Lemma 0.3, let $Y_{(y)}$ be the henselization of Y at y , and let $g_{(y)} : Y_{(y)} \rightarrow S$ be the morphism induced by g . Fix notation by the following commutative diagram:

$$\begin{array}{ccccc}
Y_{(y)} \times_k \mathbb{P}_{(\infty')}^1 & \xrightarrow{g_{(y)} \times \text{id}_{\mathbb{P}_{(\infty')}^1}} & S \times_k \mathbb{P}_{(\infty')}^1 & \xrightarrow{p_2} & \mathbb{P}_{(\infty')}^1 \\
\text{pr}_{1(y)} \downarrow & & p_1 \downarrow & & \downarrow \\
Y_{(y)} & \xrightarrow{g_{(y)}} & S & \rightarrow & \text{Spec } k.
\end{array}$$

Denote the restriction of K to $Y_{(y)}$ also by K . We have canonical morphisms

$$\begin{aligned}
&R\Psi_{\eta_{\infty'}} \left(p_1^* Rg_{(y)*} K \otimes^L \overline{\mathcal{L}_\psi(\pi t')} \right) \\
&\cong R\Psi_{\eta_{\infty'}} \left((R(g_{(y)} \times \text{id}_{\mathbb{P}_{(\infty')}^1})_* \text{pr}_{1(y)}^* K \otimes^L \overline{\mathcal{L}_\psi(\pi t')}) \right) \quad (\text{smooth base change theorem}) \\
&\cong R\Psi_{\eta_{\infty'}} \left((R(g_{(y)} \times \text{id}_{\mathbb{P}_{(\infty')}^1})_* (\text{pr}_{1(y)}^* K \otimes^L (g_{(y)} \times \text{id}_{\mathbb{P}_{(\infty')}^1})^* \overline{\mathcal{L}_\psi(\pi t')})) \right) \quad (\text{projection formula}) \\
&\rightarrow R(g_{(y)} \times \text{id}_{\infty'})_* R\Psi_{\eta_{\infty'}} \left(\text{pr}_{1(y)}^* K \otimes^L (g_{(y)} \times \text{id}_{\mathbb{P}_{(\infty')}^1})^* \overline{\mathcal{L}_\psi(\pi t')} \right)
\end{aligned}$$

Since $p_1^* Rg_{(y)*} K \otimes^L \overline{\mathcal{L}_\psi(\pi t')}$ and $\text{pr}_{1(y)}^* K \otimes^L (g_{(y)} \times \text{id}_{\mathbb{P}_{(\infty')}^1})^* \overline{\mathcal{L}_\psi(\pi t')}$ vanish on the fibers over ∞' , their vanishing cycle and nearby cycle coincides. So the composite of the above canonical morphisms can be identified with a canonical morphism

$$R\Phi_{\eta_{\infty'}} \left(p_1^* Rg_{(y)*} K \otimes^L \overline{\mathcal{L}_\psi(\pi t')} \right) \rightarrow R(g_{(y)} \times \text{id}_{\infty'})_* R\Phi_{\eta_{\infty'}} \left(\text{pr}_{1(y)}^* K \otimes^L (g_{(y)} \times \text{id}_{\mathbb{P}_{(\infty')}^1})^* \overline{\mathcal{L}_\psi(\pi t')} \right).$$

Applying Lemma 1.1 to the complex $Rg_{(y)*} K$ on S , we get a canonical isomorphism

$$R\Phi_{\eta_{\infty'}} \left(p_1^* Rg_{(y)*} K \otimes^L \overline{\mathcal{L}_\psi(\pi t')} \right) \cong R\Phi_{\eta_{\infty'}} \left(p_1^* (j_! R\Phi(Rg_{(y)*} K)) \otimes^L \overline{\mathcal{L}_\psi(\pi t')} \right).$$

By Lemma 1.2, we have

$$R\Phi(Rg_{(y)*}K) \cong (R\Phi_Y(K))_y.$$

We thus get a canonical morphism

$$R\Phi_{\eta_{\infty'}} \left(p_1^* (j_! (R\Phi_Y(K))_y) \otimes^L \overline{\mathcal{L}_\psi(\pi t')} \right) \rightarrow R(g_{(y)} \times \text{id}_{\infty'})_* R\Phi_{\eta_{\infty'}} \left(\text{pr}_1^* K \otimes^L (g_{(y)} \times \text{id}_{\mathbb{P}^1_{(\infty')}})^* \overline{\mathcal{L}_\psi(\pi t')} \right).$$

By Lemma 0.3 (i) that we have shown at the beginning of this section, this gives rise to a canonical morphism

$$R\Phi_{\eta_{\infty'}} \left(p_1^* (j_! (R\Phi_Y(K))_y) \otimes^L \overline{\mathcal{L}_\psi(\pi t')} \right)_{(s, \infty')} \rightarrow R\Phi_{\eta_{\infty'}} \left(\text{pr}_1^* K \otimes^L (g \times \text{id}_{\mathbb{P}^1_{(\infty')}})^* \overline{\mathcal{L}_\psi(\pi t')} \right)_{(y, \infty)}.$$

Under the assumption of Lemma 0.3 (iii), taking the n -th cohomology on both sides and using the definition of $\mathcal{F}^{(0, \infty')}$ in [8, 2.4.2.3], we get a canonical morphism

$$\mathcal{F}^{(0, \infty')} \left(R^{n-1} \Phi_Y(K)_y \right) \rightarrow R^n \Phi_{\eta_{\infty'}} \left(\text{pr}_1^* K \otimes^L (g \times \text{id}_{\mathbb{P}^1_{(\infty')}})^* \overline{\mathcal{L}_\psi(\pi t')} \right)_{(y, \infty')}.$$

Lemma 1.3. *Let $f : X \rightarrow S$ be a proper morphism, and let $K \in \text{ob } D_c^b(X, \overline{\mathbb{Q}}_\ell)$. Suppose f is smooth at points in $f^{-1}(s) - A$ for a finite subset A of k -rational points in $f^{-1}(s)$ and $\mathcal{H}^q(K)|_{X-A}$ are lisse for all q . Then for all $x \in A$, the canonical morphisms*

$$R\Phi_{\eta_{\infty'}} \left(p_1^* (j_! (R\Phi_X(K))_x) \otimes^L \overline{\mathcal{L}_\psi(\pi t')} \right)_{(s, \infty')} \rightarrow R\Phi_{\eta_{\infty'}} \left(\text{pr}_1^* K \otimes^L (f \times \text{id}_{\mathbb{P}^1_{(\infty')}})^* \overline{\mathcal{L}_\psi(\pi t')} \right)_{(x, \infty)}$$

constructed above are isomorphisms, where $\text{pr}_1 : X \times_k \mathbb{P}^1_{(\infty')} \rightarrow X$ is the projection. Suppose furthermore that K is a lisse sheaf and X is regular pure of dimension n . Then the canonical morphisms

$$\mathcal{F}^{(0, \infty')} \left(R^{n-1} \Phi_X(K)_x \right) \rightarrow R^n \Phi_{\eta_{\infty'}} \left(\text{pr}_1^* K \otimes^L (f \times \text{id}_{\mathbb{P}^1_{(\infty')}})^* \overline{\mathcal{L}_\psi(\pi t')} \right)_{(x, \infty')}$$

are isomorphisms for all $x \in A$.

Proof. The second statement follows from the first one. To prove the first statement, it suffices to prove the above canonical morphisms induce an isomorphism

$$\bigoplus_{x \in A} R\Phi_{\eta_{\infty'}} \left(p_1^* (j_! (R\Phi_X(K))_x) \otimes^L \overline{\mathcal{L}_\psi(\pi t')} \right)_{(s, \infty')} \rightarrow \bigoplus_{x \in A} R\Phi_{\eta_{\infty'}} \left(\text{pr}_1^* K \otimes^L (f \times \text{id}_{\mathbb{P}^1_{(\infty')}})^* \overline{\mathcal{L}_\psi(\pi t')} \right)_{(x, \infty)}.$$

Fix notation by the following commutative diagram

$$\begin{array}{ccccc} X \times_k \mathbb{P}^1_{(\infty')} & \xrightarrow{f \times \text{id}_{\mathbb{P}^1_{(\infty')}}} & S \times_k \mathbb{P}^1_{(\infty')} & \xrightarrow{p_2} & \mathbb{P}^1_{(\infty')} \\ \text{pr}_1 \downarrow & & p_1 \downarrow & & \downarrow \\ X & \xrightarrow{f} & S & \rightarrow & \text{Spec } k. \end{array}$$

We have canonical isomorphisms

$$\begin{aligned}
& R\Psi_{\eta_{\infty'}} \left(p_1^* Rf_* K \otimes^L \overline{\mathcal{L}_\psi(\pi t')} \right) \\
& \cong R\Psi_{\eta_{\infty'}} \left(R(f \times \text{id}_{\mathbb{P}^1_{(\infty')}})_* \text{pr}_1^* K \otimes^L \overline{\mathcal{L}_\psi(\pi t')} \right) \quad (\text{proper base change theorem}) \\
& \cong R\Psi_{\eta_{\infty'}} \left(R(f \times \text{id}_{\mathbb{P}^1_{(\infty')}})_* \left(\text{pr}_1^* K \otimes^L (f \times \text{id}_{\mathbb{P}^1_{(\infty')}})^* \overline{\mathcal{L}_\psi(\pi t')} \right) \right) \quad (\text{projection formula}) \\
& \cong R(f \times \text{id}_{\infty'})_* R\Psi_{\eta_{\infty'}} \left(\text{pr}_1^* K \otimes^L (f \times \text{id}_{\mathbb{P}^1_{(\infty')}})^* \overline{\mathcal{L}_\psi(\pi t')} \right) \quad (\text{proper base change theorem})
\end{aligned}$$

Since $p_1^* Rf_* K \otimes^L \overline{\mathcal{L}_\psi(\pi t')}$ and $\text{pr}_1^* K \otimes^L (f \times \text{id}_{\mathbb{P}^1_{(\infty')}})^* \overline{\mathcal{L}_\psi(\pi t')}$ vanish on the fibers over ∞' , their vanishing cycle and nearby cycle coincide. So the stalk at (s, ∞') of the composite of the above canonical isomorphisms can be identified with

$$R\Phi_{\eta_{\infty'}} \left(p_1^* Rf_* K \otimes^L \overline{\mathcal{L}_\psi(\pi t')} \right)_{(s, \infty')} \cong \left(R(f \times \text{id}_{\infty'})_* R\Phi_{\eta_{\infty'}} \left(\text{pr}_1^* K \otimes^L (f \times \text{id}_{\mathbb{P}^1_{(\infty')}})^* \overline{\mathcal{L}_\psi(\pi t')} \right) \right)_{(s, \infty')}.$$

Applying Lemma 1.1 to the complex $Rf_* K$ on S , we get

$$R\Phi_{\eta_{\infty'}} \left(p_1^* Rf_* K \otimes^L \overline{\mathcal{L}_\psi(\pi t')} \right) \cong R\Phi_{\eta_{\infty'}} \left(p_1^* j_! R\Phi(Rf_* K) \otimes^L \overline{\mathcal{L}_\psi(\pi t')} \right).$$

So we get a canonical isomorphism

$$\begin{aligned}
(6) \quad & R\Phi_{\eta_{\infty'}} \left(p_1^* j_! R\Phi(Rf_* K) \otimes^L \overline{\mathcal{L}_\psi(\pi t')} \right)_{(s, \infty')} \\
& \cong \left(R(f \times \text{id}_{\infty'})_* R\Phi_{\eta_{\infty'}} \left(\text{pr}_1^* K \otimes^L (f \times \text{id}_{\mathbb{P}^1_{(\infty')}})^* \overline{\mathcal{L}_\psi(\pi t')} \right) \right)_{(s, \infty')}.
\end{aligned}$$

By the proper base change theorem and lemma 0.1, we have

$$\begin{aligned}
R\Phi(Rf_* K) & \cong Rf_{s*}(R\Phi_X(K)) \\
& \cong \bigoplus_{x \in A} (R\Phi_X(K))_x.
\end{aligned}$$

By Lemma 0.3 (i) that we have shown at the beginning of this section, we have

$$\begin{aligned}
& \left(R(f \times \text{id}_{\infty'})_* R\Phi_{\eta_{\infty'}} \left(\text{pr}_1^* K \otimes^L (f \times \text{id}_{\mathbb{P}^1_{(\infty')}})^* \overline{\mathcal{L}_\psi(\pi t')} \right) \right)_{(s, \infty')} \\
& \cong \bigoplus_{x \in A} R\Phi_{\eta_{\infty'}} \left(\text{pr}_1^* K \otimes^L (f \times \text{id}_{\mathbb{P}^1_{(\infty')}})^* \overline{\mathcal{L}_\psi(\pi t')} \right)_{(x, \infty')}.
\end{aligned}$$

So we can write the isomorphism (6) as

$$\begin{aligned}
& \bigoplus_{x \in A} R\Phi_{\eta_{\infty'}} \left(p_1^* j_! (R\Phi_X(K))_x \otimes^L \overline{\mathcal{L}_\psi(\pi t')} \right)_{(s, \infty')} \\
& \cong \bigoplus_{x \in A} R\Phi_{\eta_{\infty'}} \left(\text{pr}_1^* K \otimes^L (f \times \text{id}_{\mathbb{P}^1_{(\infty')}})^* \overline{\mathcal{L}_\psi(\pi t')} \right)_{(x, \infty')}.
\end{aligned}$$

This last isomorphism coincides with the morphism induced by the canonical morphisms

$$R\Phi_{\eta_{\infty'}} \left(p_1^* j_! (R\Phi_X(K))_x \otimes^L \overline{\mathcal{L}_\psi(\pi t')} \right)_{(s, \infty')} \rightarrow R\Phi_{\eta_{\infty'}} \left(\text{pr}_1^* K \otimes^L (f \times \text{id}_{\mathbb{P}^1_{(\infty')}})^* \overline{\mathcal{L}_\psi(\pi t')} \right)_{(x, \infty')}.$$

It follows that these canonical morphisms are isomorphisms. \square

We are now ready to prove Lemma 0.3 (ii) and (iv).

Proof of Lemma 0.3 (ii) and (iv). The canonical morphism

$$R\Phi_{\eta_{\infty'}} \left(p_1^* (j_! (R\Phi_Y(K))_y) \otimes^L \overline{\mathcal{L}_\psi(\pi t')} \right)_{(s, \infty')} \rightarrow R\Phi_{\eta_{\infty'}} \left(\text{pr}_1^* K \otimes^L (g \times \text{id}_{\mathbb{P}^1_{(\infty')}})^* \overline{\mathcal{L}_\psi(\pi t')} \right)_{(y, \infty')}$$

commutes with the base change on the trait S . Using [4, 3.7], we can reduce to the case where k is algebraically closed and $S = \text{Spec } k[[\pi]]$. The problem is local at y with respect to the étale topology. By [6, 2.5], we may assume that Y is an open subscheme of a scheme X , $g : Y \rightarrow S$ can be extended to a proper morphism $f : X \rightarrow S$ which is smooth at points in $f^{-1}(0) - A$ for a finite subset A of closed points in $f^{-1}(0)$, $y \in A$, (and any $x \in A - \{y\}$ is an ordinary quadratic singular point of $f^{-1}(0)$, a condition that we will not use). Our assertion then follows from Lemma 1.3. \square

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